10. Absolute continuity and singularity

How wild can the jumps of a CDF be? If μ is a p.m. on $\mathbb{R}$ with CD F that has a jump at x, that means $\mu x = F(x) - F(x-) > 0$. Since the total probability is one, there can be atmost n jumps of size $\frac{1}{n}$ or more. Putting them together, there can be atmost countably many jumps. In particular F is continuous on a dense set. Let J be the set of all jumps of F. Then, $F = F_{\text{atom}} + F_{\text{cts}}$ where $F_{\text{atom}}(x) := \sum_{x \in J}(F(x) - F(x-))$ and $F_{\text{cts}} = F - F_{\text{atom}}$. Clearly, F_{cts} is a continuous non-decreasing function, while F_{atom} is a non-decreasing continuous function that increases only in jumps (if $J \cap [a,b] = \emptyset$, then $F_{\text{atom}}(a) = F_{\text{atom}}(b)$).

If F_{atom} is not identically zero, then we can scale it up by $c = (F_{\text{atom}}(+\infty) - F_{\text{atom}}(-\infty))^{-1}$ to make it a CDF of a p.m. on $\mathbb{R}$. Similarly for F_{cts}. This means, we can write μ as $c\mu_{\text{atom}} + (1-c)\mu_{\text{cts}}$ where $c \in [0,1]$ and μ_{atom} is a purely atomic measure (its CDF increases only in jumps) and μ_{cts} has a continuous CDF.

Definition 37. Two measures μ and ν on the same $(\Omega, \mathcal{F})$ are said to be *mutually singular* and write $\mu \perp \nu$ if there is a set $A \in \mathcal{F}$ such that $\mu(A) = 0$ and $\nu(A^c) = 0$. We say that μ is *absolutely continuous to* ν and write $\mu \ll \mu$ if $\mu(A) = 0$ whenever $\nu(A) = 0$.

Remark 38. (i) Singularity is reflexive, absolute continuity is not. If $\mu \ll \nu$ and $\nu \ll \mu$, then we say that μ and ν are *mutually absolutely continuous*. (ii) If $\mu \perp \nu$, then we cannot also have $\mu \ll \nu$ (unless $\mu = 0$). (iii) Given μ and ν, it is not necessary that they be singular or absolutely continuous to one another.

Example 39. Uniform($[0,1]$) and Uniform($[1,2]$) are singular. Uniform($[1,3]$) is neither absolutely continuous nor singular to Uniform($[2,4]$). Uniform($[1,2]$) is absolutely continuous to Uniform($[0,4]$) but not conversely. All these uniforms are absolutely continuous to Lebesgue measure. Any measure on the line that has an atom (eg., δ_0) is singular to Lebesgue measure. A p.m. on the line with density (eg., $N(0,1)$) is absolutely continuous to $\mathbf{m}$. In fact $N(0,1)$ and $\mathbf{m}$ are mutually absolutely continuous. However, the exponential distribution is absolutely continuous to Lebesgue measure, but not conversely (since $(-\infty, 0)$, has zero probability under the exponential distribution but has positive Lebesgue measure).

As explained above, a p.m on the line with atoms is singular (w.r.t $\mathbf{m}$). This raises the natural question of whether every p.m. with a continuous CDF is absolutely continuous to Lebesgue measure? Surprisingly, the answer is No!

Example 40 (Cantor measure). Let K be the middle-thirds Cantor set. Consider the canonical probability space $([0,1], \mathcal{B}, \mathbf{m})$ and the random variable $X(\omega) = \sum_{k=1}^{\infty} \frac{2X_k(\omega)}{3^k}$, where $X_k(\omega)$ is the k^{th} binary digit of ω (i.e., $\omega = \sum_{k=1}^{\infty} \frac{X_k(\omega)}{2^k}$). Then X is measurable (why?). Let $\mu := \mathbf{m}X^{-1}$ be the pushforward.

Then, $\mu(K) = 1$, because X takes values in numbers whose ternary expansion has no ones. Further, for any $t \in K$, $X^{-1}\{t\}$ is a set with atmost two points and hence has zero Lebsgue measure. Thus μ has not atoms and must have a continuous CDF. Since $\mu(K) = 1$ but $\mathbf{m}(K) = 0$, we also see that $\mu \perp \mathbf{m}$.

Exercise 41 (Alternate construction of Cantor measure). Let $K_1 = [0,1/3] \cup [2/3,1]$, $K_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$, etc., be the decreasing sequence of compact sets whose intersection is K. Observe that K_n is a union of 2^n intervals each of length 3^{-n}. Let μ_n be the p.m. which is the "renormalized Lebesgue measure" on K_n. That is, $\mu_n(A) := 3^n 2^{-n}\mathbf{m}(A \cap K_n)$. Then each μ_n is a Borel p.m. Show that $\mu_n \xrightarrow{d} \mu$, the Cantor measure.

Example 42 (Bernoulli convolutions). We generalize the previous example. For any $\lambda > 1$, define $X_\lambda : [0,1] \to \mathbb{R}$ by $X(\omega) = \sum_{k=1}^{\infty} \lambda^{-k} X_k(\omega)$. Let $\mu_\lambda = \mathbf{m}X_\lambda^{-1}$ (did you check that X_λ is measurable?). For $\lambda = 3$, this is almost the same as $1/3$-Cantor measure, except that we have left out the irrelevant factor of 2 (so μ_3 is a p.m. on $\frac{1}{2}K := \{x/2 \,:\, x \in K\}$) and hence is singular.

Exercise 43. For any $\lambda > 2$, show that μ_λ is singular w.r.t. Lebesgue measure.

For $\lambda = 2$, it is easy to see that μ_λ is just the Lebesgue measue on $[0,1/2]$. Hence, one might expect that μ_λ is absolutely continuous to Lebesgue measure for $1 < \lambda < 2$. This is false! Paul Erdős showed that μ_λ is singular to Lebesgue measure whenever λ is a Pisot-Vijayaraghavan number, i.e., if λ is an algebraic number all of whose conjugates have modulus less than one!! It is an open question as to whether these are the only exceptions.

Theorem 44 (Radon Nikodym theorem). *Suppose μ and ν are two measures on $(\Omega, \mathcal{F})$. Then $\mu \ll \nu$ if and only if there exists a non-negative measurable function $f : \Omega \to [0, \infty]$ such that $\mu(A) = \int_A f(x)d\nu(x)$ for all $A \in \mathcal{F}$.*

Remark 45. Then, f is called the *density of μ with respect to* ν. Note that the statement of the theorem does not make sense because we have not defined what $\int_A f(x)d\nu(x)$ means! That will come next class, and then, one of the two implications of the theorem, namely, "if μ has a density w.r.t. μ, then $\mu \ll \nu$" would become obvious. The converse statement, called the Radon-Nikodym theorem is non-trivial and will be proved in the measure theory class.